NORTHWATER CAPITAL MANAGEMENT INC.

Marginal Contribution to the Sharpe Ratio

Simon Chan
Adrian Hussey

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Simon Chan is Vice President, Northwater Capital Management Inc.
Adrian Hussey is Vice President, Northwater Capital Management Inc.

In this paper, we provide a framework for interpreting the marginal impact of adding an asset to a portfolio. Using this framework, we define the condition for an asset to be a positive contributor to the portfolio Sharpe ratio. Also defined is the condition for multiple assets to contribute equally to the portfolio Sharpe ratio. Using the marginal contribution framework, the implied return framework is introduced where a manager's view is presented as the quantity necessary to bring a portfolio back to its equilibrium state. The framework proposed by this paper is not changed by the version of Sharpe ratio used. In fact, the analysis that follows is portable to other risk return measures such as the Sortino ratio, the Information ratio, the Excess Return on VaR, the Conditional Sharpe ratio and the Modified Sharpe ratio, among others. A numerical example is included as illustration, where we demonstrated how an asset manager can apply the framework to make incremental allocation decisions towards an optimal portfolio.

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Introduction

In the early days of investment management, portfolio managers had vague ideas about risk and its role in portfolio construction. The concept of a risk return trade-off was not defined until the publication of Harry Markowitz's seminal paper titled "Portfolio Selection" in 1952. In this study, Harry Markowitz introduced the concepts of a portfolio model where he defined an efficient portfolio as one that maximizes mean return for a given level of variance. The introduction of this concept to the field of asset management revolutionized thinking and led to the introduction of what became known as the Sharpe ratio by William Sharpe in 1966. Under Markowitz, the market portfolio has the highest Sharpe ratio and corresponds to the portfolio where the Capital Allocation Line is tangent to the efficient frontier. To this day, the Sharpe ratio remains one of the most cited and widely used measures in financial analysis. This statistic has been widely accepted and used by the investment community to help make asset allocation decisions across all asset classes and as a quantitative measure of performance.

However, little attention has been given to the Sharpe ratio on a marginal portfolio contribution context. More precisely, what is the impact to the portfolio's Sharpe ratio when an asset is being added to the mix? In this paper, we provide a framework for interpreting the marginal impact of adding an asset to a portfolio. Using this framework, we define the condition for an asset to be a positive contributor to the portfolio Sharpe ratio. We also examine the risk return trade-off in different market conditions and illustrate conditions under which a Sharpe maximizing portfolio manager is positioned to look for returns or favour diversification benefits. Using the marginal contribution framework, the implied return framework is introduced where a manager's view is presented as the quantity necessary to bring a portfolio back to its equilibrium state.

The Sharpe Ratio

There are numerous variations of the Sharpe ratio presented in the finance literature and used by practitioners. However, they all provide a measure of reward per unit of risk that boils down to the ratio of an excess return relative to the risk-free rate over a measure of risk. Opdyke (2006) gave a good summary of their computations and interpretations. The framework proposed by this paper is not changed by the version of Sharpe ratio used. In fact, the analysis that follows is portable to other risk return measures such as the Sortino ratio, the Information ratio, the Excess Return on VaR, the Conditional Sharpe ratio and the Modified Sharpe ratio, among others. In order to calculate confidence interval for Sharpe ratio, we make the same assumption as Opdyke that risk-free rate is constant over the measurement period.

Marginal Contribution Framework

The framework calls for viewing return, risk and the Sharpe ratio from the marginal contribution perspective. The marginal contribution measure is defined as the change in the measure when a small portion (say 1%) of the asset in question is added to the existing portfolio. In other words, the marginal contribution framework always links the portfolio to an asset, which is consistent with Markowitz. The candidate asset can be a fund, a representative strategy or even another portfolio.

The marginal contribution to return is the change in portfolio return when a small portion of the asset is added to the portfolio. Since portfolio return is linear in its constituent assets, the addition of an asset with higher return than the portfolio will result in an increase in average portfolio return and vice versa.

The marginal contribution to risk is the change in portfolio standard deviation of the asset is added to the portfolio. It is well known that an investor can reduce portfolio risk by holding instruments which are not perfectly correlated. The marginal contribution of an asset to portfolio risk is therefore driven by the risk of the asset, the risk of the portfolio and the asset's correlation to the portfolio. Obviously, a lower risk asset will be risk reducing, but a higher risk asset can also be risk reducing as long as its diversification (correlation) benefit outweighs its risk (volatility) contribution.

Lastly, the marginal contribution to the Sharpe ratio is the change in portfolio Sharpe ratio when a small portion of the asset is added to the portfolio. This marginal effect links together the return and risk trade-off captured by the Sharpe ratio as well as the portfolio diversification characteristic of an asset in the Markowitz framework. In what follows, we present the relationships that tie these marginal contributions together and some practical application in portfolio construction.

Optimal Sharpe Line (OSL)

All asset managers attempt to optimize the return and risk trade-off for their clients so that the allocated capital can attain the maximum growth for an appropriate target level of risk. In practice, asset managers are constantly faced with asset allocation decisions as information becomes available. It is important to keep track of how each asset is contributing to the return and risk trade off as revisions are being made to expectations.

It can be shown[1] that for small allocation changes, the marginal contribution to the portfolio Sharpe ratio depends only on the marginal contribution to return, the marginal contribution to risk, the portfolio Sharpe ratio and the current portfolio risk.

$$\Delta r = \sigma \, \Delta S + (S + \Delta S) \, \Delta\sigma$$

This equation represents a straight line where the x-axis is the marginal contribution to risk ($\Delta\sigma$), the y-axis is the marginal contribution to return (Δr), the y-intercept is the product of portfolio risk and marginal contribution to portfolio Sharpe ratio ($\sigma\Delta S$). The slope of the line is the portfolio Sharpe ratio ($S+\Delta S$).

By plotting this equation, we obtain what we define here as the Optimal Sharpe Line (OSL) by setting ΔS to 0, which relates the marginal contribution to return and the marginal contribution to risk by the portfolio Sharpe ratio. Figure 1 shows the Optimal Sharpe Line. By definition, an asset that falls on the OSL has a marginal contribution of zero to the portfolio Sharpe ratio. This implies an asset above the OSL has positive marginal contribution to portfolio Sharpe ratio and vice versa.

[1] The appendix gives the derivation of the Optimal Sharpe Line

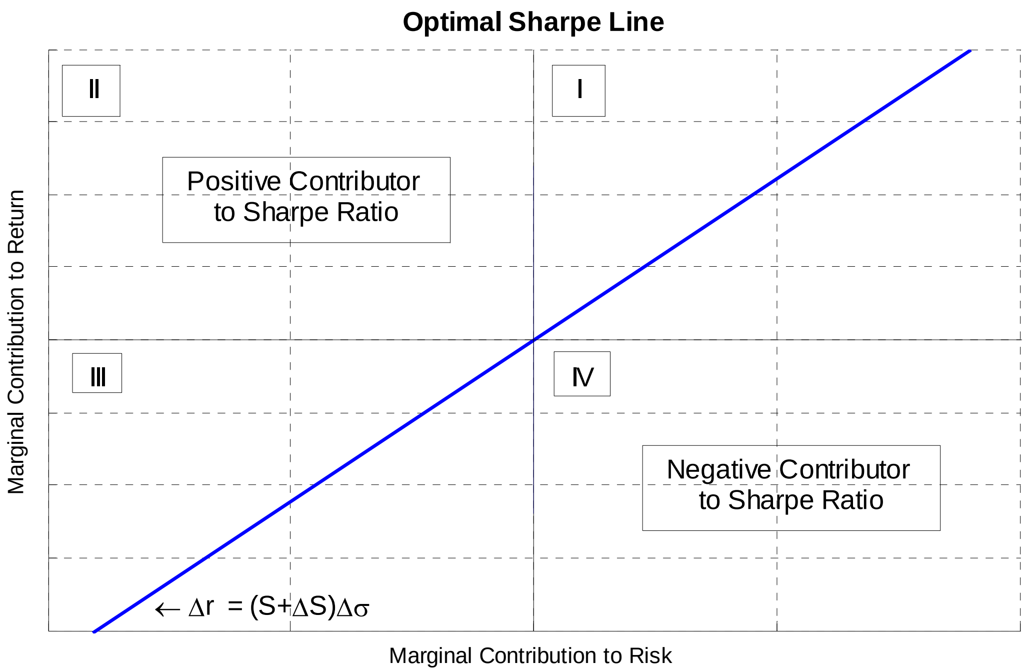

Figure 1: Optimal Sharpe Line (OSL)

The OSL always has a slope equal to the Sharpe ratio of the portfolio, and the portfolio itself always lies on the OSL. The OSL of a portfolio with positive Sharpe ratio will always cut across quadrants I and III and also across the portfolio itself. This implies that quadrant II assets are always positive contributors whereas quadrant IV assets are always negative contributors to the portfolio Sharpe ratio. Quadrant I and III assets can contribute positively[2] or negatively[3] to the portfolio Sharpe ratio depending on the current value of the portfolio Sharpe ratio. Table 1 summarizes these relationships.

Quadrant	Contribution to Return	Contribution to Risk	Contribution to Sharpe Ratio
I	Bigger than Portfolio	Bigger than Portfolio	Depends on portfolio Sharpe
II	Bigger than Portfolio	Smaller than Portfolio	Increase
III	Smaller than Portfolio	Smaller than Portfolio	Depends on portfolio Sharpe
IV	Smaller than Portfolio	Bigger than Portfolio	Decrease

Table 1: Positive and negative contributors to portfolio Sharpe ratio

Specifically, in a low portfolio Sharpe regime, where excess return is poor and/or risk incurred is high, quadrant I assets are likely to be portfolio Sharpe ratio enhancers while quadrant III assets are likely to be portfolio Sharpe ratio reducers. Alternatively, in a high portfolio Sharpe regime, where excess return is high and/or risk incurred is low, quadrant I assets are likely to be portfolio Sharpe ratio reducers while quadrant III assets are likely to be portfolio Sharpe ratio enhancers.

[2] The condition for an asset to contribute positively to portfolio Sharpe ratio is: $\Delta r > (S+\Delta S)\,\Delta\sigma$
[3] The condition for an asset to contribute negatively to portfolio Sharpe ratio is: $\Delta r < (S+\Delta S)\,\Delta\sigma$

In the extreme, when the portfolio Sharpe ratio is 0, any asset that delivers contribution to portfolio return higher than the portfolio return will increase portfolio Sharpe ratio, regardless of its risk contribution. In a recent survey[4] of finance executives, 76% of the respondents said their company planned on risk reduction in pension portfolios at the expense of return on assets. Our findings suggest that asset managers should instead focus on the search for return enhancing assets during tough economic times in order to improve their portfolio's Sharpe ratio. That being said, the risk dimension should be carefully managed as when the portfolio Sharpe ratio starts to improve, assets that add to risk will contribute negatively to portfolio Sharpe ratio if its return contribution is not high enough.

The converse is also true. When the portfolio Sharpe ratio is high, any asset that reduces portfolio risk will increase the portfolio Sharpe ratio regardless of its return contribution. This implies that asset managers should focus on the search for diversifying assets during upswings of economic cycles in order to improve their portfolio return versus risk trade-off.

FORMING AN OPTIMAL SHARPE PORTFOLIO

Suppose an asset manager has an existing portfolio and his goal is to select assets that optimize the portfolio Sharpe ratio. He can do so by calculating the marginal return and risk contribution of the assets to his existing portfolio and plotting them against the Optimal Sharpe Line (OSL). Figure 2 shows a plot of some representative assets in relation to an OSL.

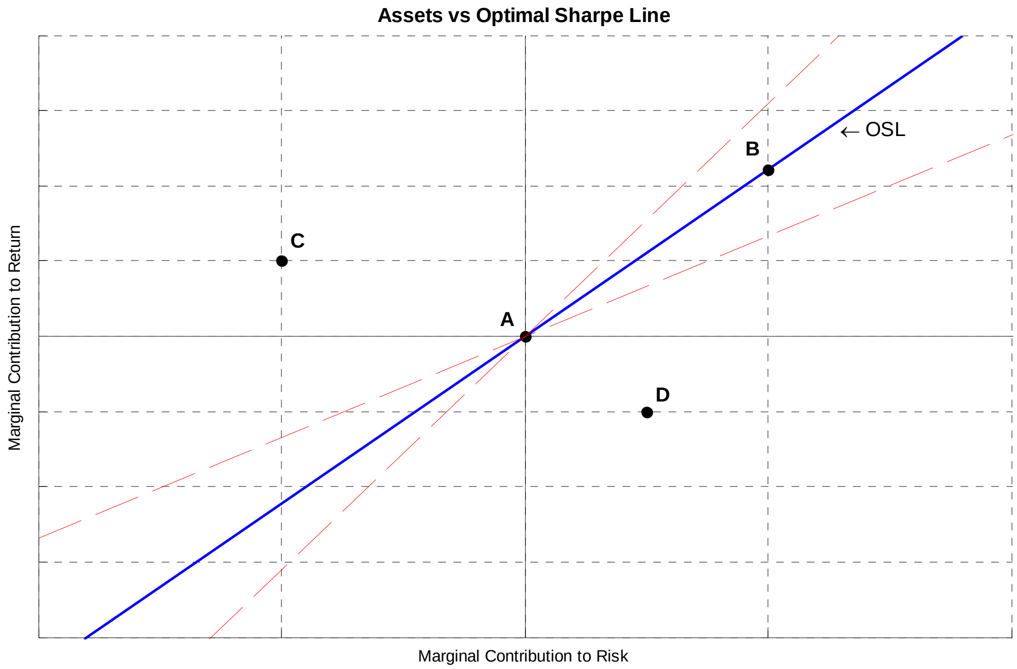

Figure 2: Assets relative to Optimal Sharpe Line

4 CFO Research Services in collaboration with Towers Perrin surveyed senior finance executives' view on their company's pension plans in the US and Canada, published in September 2008.

The two dotted lines are the upper OSL and the lower OSL constructed using the confidence interval of the portfolio Sharpe ratio. Assets that fall in between the two dotted lines do not make significant marginal contributions to the portfolio Sharpe ratio.

Asset A is the portfolio itself. In equilibrium, all assets in the portfolio should lie on the OSL (or in between the upper OSL and the lower OSL). Asset B is an asset in equilibrium. The manager can allocate towards Asset C to achieve a higher portfolio Sharpe ratio since it is above the OSL. Similarly, allocation to Asset D can be reduced to achieve a higher portfolio Sharpe ratio. Modifying the portfolio in this manner iteratively will obtain convergence to the portfolio with the highest Sharpe ratio (or the efficient portfolio in Markowitz's world).

Hence, the marginal contribution framework exposes information otherwise hidden in the traditional Markowitz portfolio optimizer. By examining the marginal statistics of an asset class relative to an intermediate portfolio, analyst can gain insight into how the optimizer chooses to allocate towards one asset but not others. Asset managers can identify the allocation sweet spot by employing a technique where successive turnover of the portfolio is simulated and allocated towards the Markowitz efficient portfolio. This technique is demonstrated in the numerical example in a later section.

Implied Return Framework

Asset managers may not implement the equilibrium portfolio for a variety of reasons. The asset manager may have a view on an asset that causes him to deviate from the equilibrium. Our model allows for the quantification of this view in the return space, which we called implied return. Holding the risk dimension constant and for small changes in allocation, the implied return of the asset is the expected asset return that would put an asset on the OSL. It is the asset return minus the vertical distance[5] between the asset and the OSL. This is in spirit similar to the approach of Sharpe (1974) where he obtained expected returns from portfolio holdings. The implied return obtained this way can be used by the asset manager to make relative comparisons among her assets as it puts all asset allocation decisions back on the return space.

When an asset is on the OSL, the implied return equals the asset return as the asset is already in equilibrium. However, if at equilibrium an asset is not on the OSL, its implied return should reflect the manager's view, which (if realized) would effectively put the asset back on the OSL. An underweight asset such as Asset C should have an implied return that is lower than its expected return, while an overweight asset such as Asset D should have an implied return that is higher that its expected return. Table 2 summarizes these relationships.

Asset	Type	Allocation Impact	Implied Asset Return
A	Portfolio	N/A	N/A
B	Equilibrium	None	Same as expected
C	Underweight	Increases Sharpe ratio	Lower than expected
D	Overweight	Deceases Sharpe ratio	Higher than expected

Table 2: Interpreting assets that are on or off the OSL

[5] Implied return can be obtained by the equation: $IR = S\ \Delta\sigma / \Delta w = r - (\sigma\ \Delta S / \Delta w + \sigma\ \Delta S / \Delta p)$

INDIFFERENCE SHARPE LINES (ISL)

We now extend the Optimal Sharpe Line (OSL) to a set of Indifference Sharpe Lines (ISLs). Each ISL represents the same marginal contribution to the portfolio Sharpe ratio. That is, when two assets fall on the same ISL, the portfolio is indifferent between these assets as they have the same marginal contribution to the portfolio Sharpe ratio. This is the case of moving along an ISL.

In Figure 3, Asset E has diversification benefit while Asset F adds to portfolio risk. Hence, in order to deliver the same marginal benefit to the portfolio Sharpe ratio, Asset F needs to deliver higher marginal return. More precisely, the additional marginal benefit to return required from an asset, given the difference in marginal contribution to risk needed for the asset to deliver the same marginal benefit in the portfolio Sharpe ratio, is the product of the new Sharpe ratio and the difference in marginal contribution to risk. This quantity is represented by the vertical distance[6] between Asset E and Asset F in Figure 3.

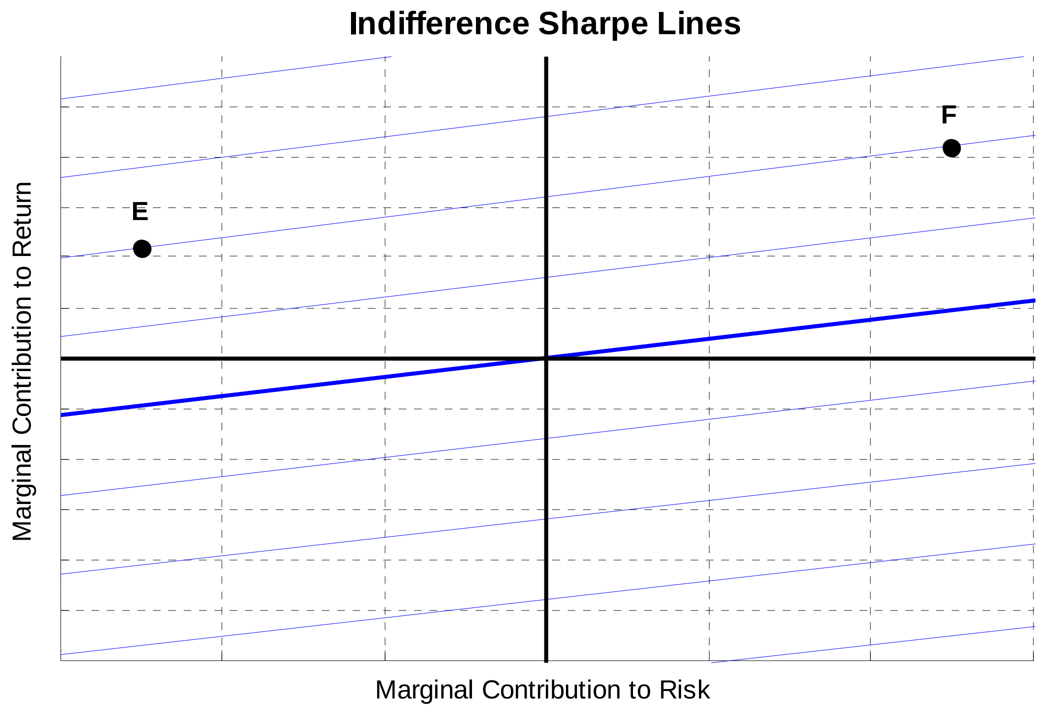

Figure 3: Indifference Sharpe Lines (ISLs)

When moving from one ISL to another along the risk dimension[7], a higher risk portfolio requires an asset to deliver a higher marginal contribution to return in order to have the same improvement in the portfolio Sharpe ratio.

The ISLs are handy for an asset manager to visualize his assets contributions to the portfolio's risk, return and Sharpe ratio relative to each other. The framework is flexible enough to handle the manager's choice of the definition of risk in order to handle risk characteristics arising from specific asset classes.

[6] Along any ISL, we have $\Delta r = \sigma \, \Delta S + (S + \Delta S)\, \Delta\sigma$. Hence, $\Delta r_1 - \Delta r_2 = (S + \Delta S)\,(\Delta\sigma_1 - \Delta\sigma_2)$.

[7] Moving from one ISL to another, we have the relationship: $\Delta S_1 - \Delta S_2 = (\Delta r_1 - \Delta r_2) / (\sigma + \Delta\sigma)$

Numerical Example[8]

An asset manager is looking for ways to enhance her portfolio return while keeping risk low. Consider the case of a manager with a portfolio consisting of 100% US equity who is evaluating the impact of adding US bond, foreign equity, real estate, commodities and alternative beta[9][10] to his portfolio. Table 3 shows the long-term estimates using historical returns[11] from 1972 – 2008.

Statistic	US T-Bills	US Equity	US Bond	EAFE	Real Estate	Commo-dities	Alt. Beta
Return (%)	5.91	11.28	8.41	12.23	12.98	12.77	11.86
Risk (%)	2.96	19.06	6.32	23.31	19.15	26.24	7.8
Sharpe Ratio		0.28	0.40	0.27	0.37	0.26	0.76

Table 3: Annual return and risk statistics (1972 - 2008)

The existing equity portfolio has a Sharpe ratio of 0.28. Closer inspection reveals that US bond, real estate and alternative beta has a higher Sharpe ratio than the portfolio, implying that adding US bond, real estate or alternative beta should have a positive contribution to the portfolio Sharpe ratio. Commodities and EAFE both have high returns but the associated high risks drive their Sharpe ratio down. US Bond has the lowest return and risk statistics. Focusing on the individual Sharpe ratios of the asset classes alone, the asset manager would not allocate towards EAFE and Commodities.

Applying the marginal contribution[12] framework results in Table 4 and Figure 4 where the statistics represent the effect of adding 1% of an asset to the portfolio.

Marginal Contribution	US Equity (Portfolio)	US Bond	EAFE	Real Estate	Commo-dities	Alt. Beta
Return (bps)	11.28	8.41	12.23	12.98	12.77	11.86
Risk (bps)	19.06	0.95	15.49	11.79	(1.92)	1.45
Sharpe Ratio	0.0031	0.0043	0.0041	0.0051	0.0070	0.0060

Table 4: Marginal contribution of adding 1% of an asset to the portfolio

Note how we illustrate the marginal contribution framework in this numerical example by incrementing each asset weight by 1%. At the limit when delta weight goes to zero, the OSL in Figure 4 will pass through the portfolio at (0, 0).

[8] The numerical example is included to illustrate concepts presented in the paper and do not constitute investment recommendations
[9] Northwater Capital Management has investigated the risk and return characteristics of various alternative betas and have created a portfolio using individual alternative beta premia as building blocks.
[10] We define "alternative beta" as a risk premia which has an expected positive excess return over moderate time horizons. Similar to a traditional risk premia such as the equity risk premium, there is a structural reason for the existence and persistence of positive excess return from each alternative beta premia.
[11] Source: Bloomberg, Northwater Capital Management
[12] Rearranging the equation for OSL we can obtain marginal contribution to Sharpe ratio as: $\Delta S = (\Delta r - S\Delta\sigma) / (\sigma + \Delta\sigma)$

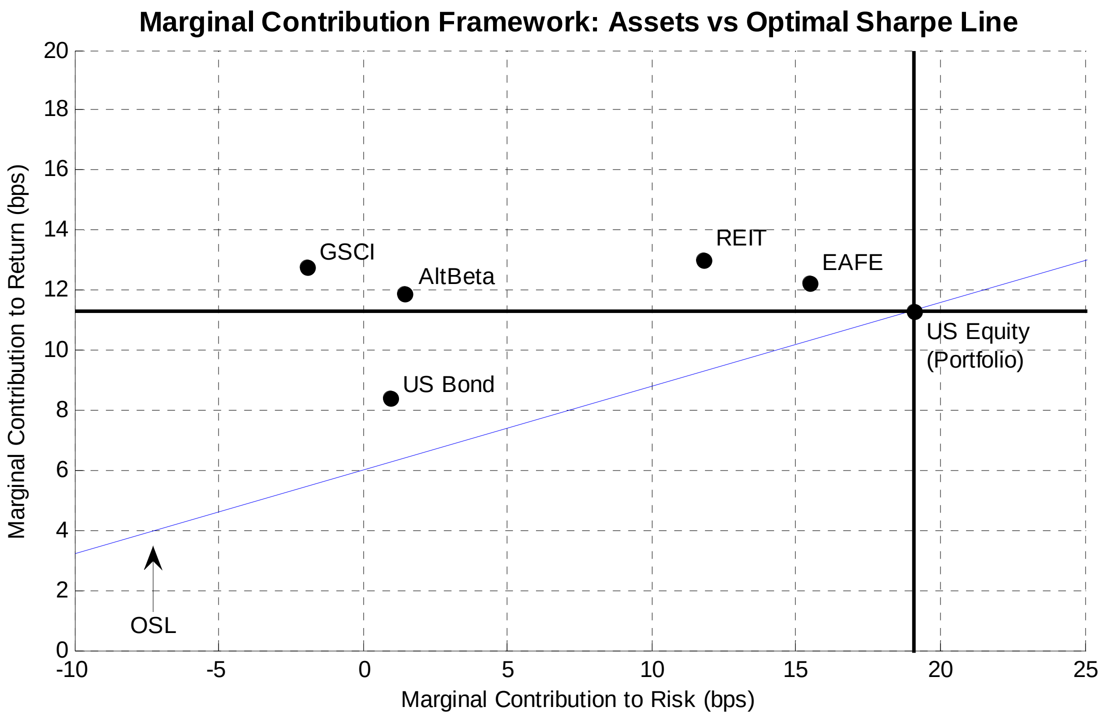

Figure 4: Commodity, real estate and foreign equity relative to US Equity (Portfolio)

The slope of the OSL is the Sharpe ratio of the equity portfolio which is 0.28. Commodity, alternative beta, real estate and foreign equity all sit in quadrant II indicating they all add to the Sharpe ratio when added to the portfolio. Commodities have the best marginal contribution to the portfolio Sharpe ratio of 0.0070, followed by alternative beta of 0.0060 and then real estate of 0.0051. Although commodities have the highest risk on a stand alone basis, they have the lowest marginal risk contribution to the portfolio due to their negative correlation to US Equity.

Table 5 and Figure 5 summarized the results by asset class obtained by applying the implied return framework.

	US Equity (Portfolio)	US Bond	EAFE	Real Estate	Commo-dities	Alt. Beta
Historical Return (%)	11.28	8.41	12.23	12.98	12.77	11.86
Implied Return (%)	11.28	6.18	10.28	9.23	5.37	6.32

Table 5: Historical vs. implied equilibrium return

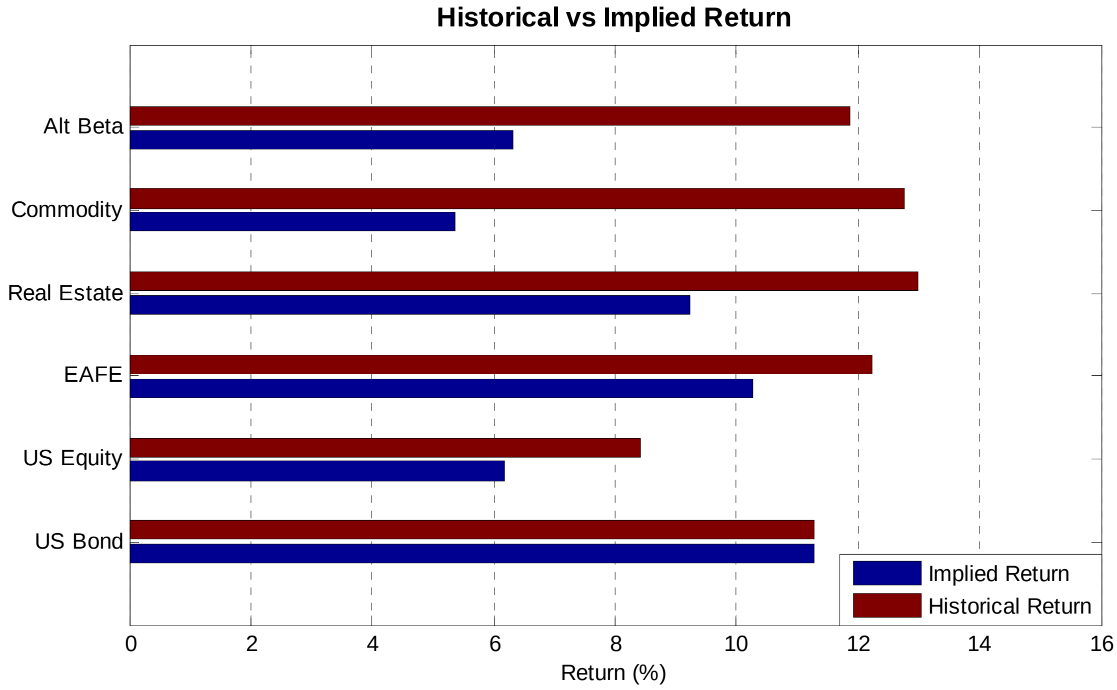

Figure 5: Historical vs. implied equilibrium return

The implied return approach quantifies the qualitative aspects of the asset allocation process into a single return dimension. Commodities returned 13% annually for the 37 year period from 1972 – 2008. This asset class lies the furthest away vertically above the OSL in Figure 4, meaning that it is severely underweight in the portfolio. Its implied return of 5% suggests that the asset manager holding a portfolio of US equities who would not allocate towards commodities has a collective view that commodities will return 5% or less in the next year. Similarly, alternative beta historically returned 12% annually. Its implied return of 6% suggests that the asset manager holding a portfolio of US equities who would not allocate towards alternative beta has a collective view that alternative beta will return 6% or less in the next year.

The natural next step would be to start allocating capital to underweight asset classes and vice versa. While the Markowitz efficient portfolio is an obvious target portfolio, it is often a hurdle for an asset manager to make a drastic change in the portfolio composition. The marginal contribution framework can be used as a platform where an asset manager can evaluate the impact of an allocation change to the portfolio.

Figure 6 shows the transition diagram and the Sharpe ratio of the portfolio as it converges to the Markowitz efficient portfolio. Figure 7 is the same convergence plotted against the Markowitz efficient frontier. In these figures, each of the iterations represents a fixed 4% turnover of the portfolio, where assets with negative marginal contribution to the portfolio Sharpe ratio are liquidated and the proceeds allocated to assets with positive marginal contribution to the portfolio Sharpe ratio.

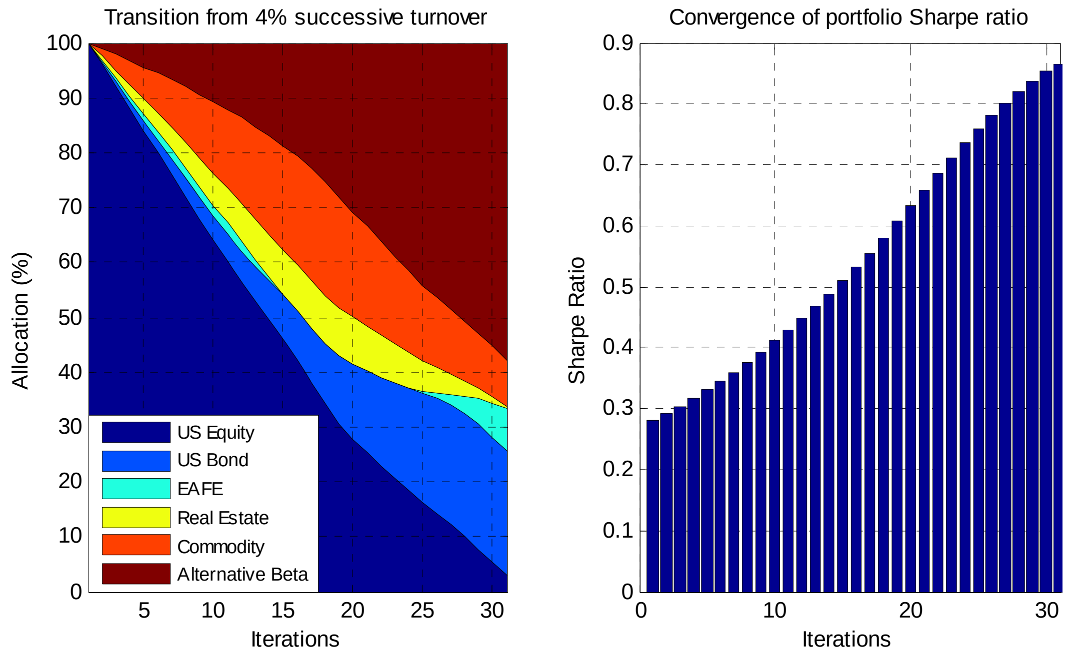

Figure 6: Transition diagram and convergence of the portfolio Sharpe ratio

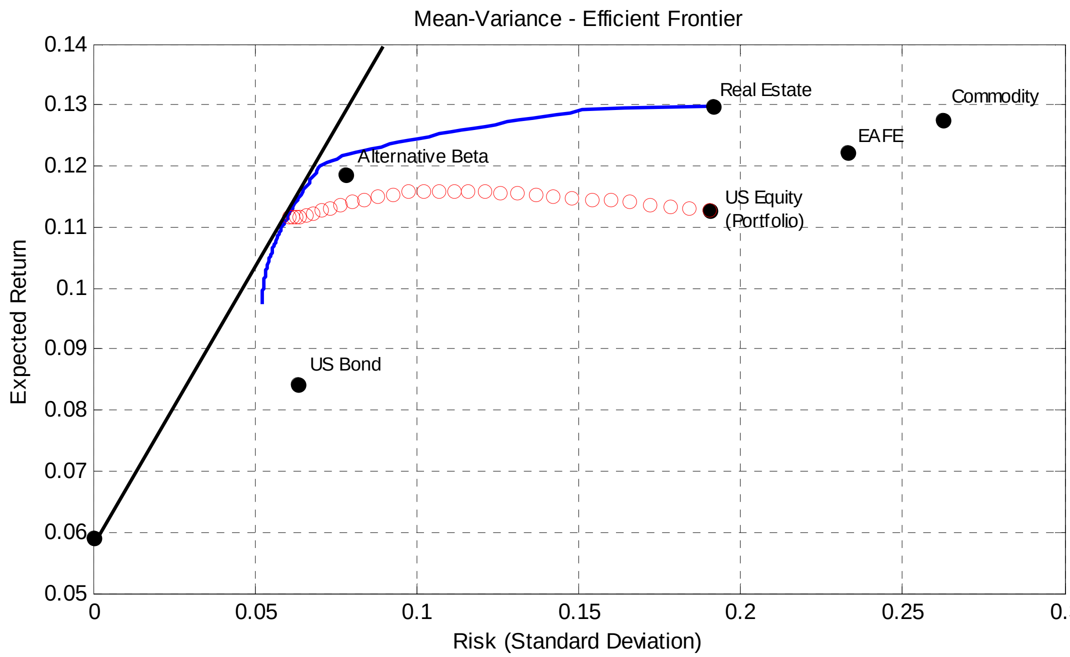

Figure 7: Portfolio convergence against the Markowitz efficient frontier

The portfolio Sharpe ratio convergence plot in Figure 6 has the shape of an "S" curve, which implies increasing marginal benefit from turning over the portfolio for the first half of the iterations and decreasing marginal benefit from turning over the portfolio for the second half of the iterations. The allocation sweet spot in this example is around half way between the current portfolio and the Markowitz efficient portfolio.

It is possible to have other convergence patterns and a trade off always exist between turning over the portfolio vs. the expected gain. By knowing the characteristics of the candidates intermediate to the current portfolio and the efficient portfolio, the asset manager can make better informed allocation recommendations to the investment committee.

CONCLUSION

In this paper we presented a framework where the marginal contributions of an asset on the portfolio's return, risk and Sharpe ratio can be interpreted together. The Optimal Sharpe Line is defined and an asset's positioning relative to the OSL is analysed. The concept of the implied return is used to reconcile the deviation of an asset manager's portfolio from the equilibrium portfolio. The Optimal Sharpe Line concept is extended to the Indifference Sharpe Lines.

The OSL and the ISLs plotted on an asset map are useful in helping make incremental asset allocation decisions as well as reconciling qualitative views to current allocations. We looked at a hypothetical example of how an asset manager can apply the framework to analyze her allocation decisions and obtain the allocation sweet spot.

APPENDIX – OPTIMAL SHARPE LINE, INDIFFERENCE SHARPE LINES AND IMPLIED RETURN DERIVED

The Sharpe ratio S is defined as,

$$S = \frac{x^T \mu - r_f}{(x^T V x)^{1/2}} \qquad (1)$$

where,

r_f is the risk free rate, $\boldsymbol{x}$ are the asset weights, $\boldsymbol{\mu}$ are the expected returns and $\boldsymbol{V}$ is the variance-covariance matrix where σ_{ii} is the variance of asset i and σ_{ij} is the covariance of asset i and asset j. r and σ are the portfolio return and portfolio risk respectively, T is the transpose operator.

$$x = \begin{bmatrix} x_1 \\ \bullet \\ \bullet \\ x_n \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \bullet \\ \bullet \\ \mu_n \end{bmatrix}, \quad V = \begin{bmatrix} \sigma_{11} & \bullet & \bullet & \sigma_{1n} \\ \bullet & \bullet & & \bullet \\ \bullet & & \bullet & \bullet \\ \sigma_{n1} & \bullet & \bullet & \sigma_{nn} \end{bmatrix} \quad \text{and} \quad r = x^T \mu \,, \quad \sigma = (x^T V x)^{1/2}$$

Rewriting equation (1) for S as,

$$S \times (x^T V x)^{1/2} = x^T \mu - r_f \qquad (2)$$

Differentiate both sides of (2) with respect to $\boldsymbol{x}$,

$$\frac{d}{dx} S \times (x^T V x)^{1/2} = \frac{d}{dx}(x^T \mu - r_f)$$

$$S \frac{d}{dx}(x^T V x)^{1/2} + (x^T V x)^{1/2} \frac{dS}{dx} = \frac{d}{dx}(x^T \mu)$$

$$S \frac{Vx}{(x^T V x)^{1/2}} + (x^T V x)^{1/2} \frac{dS}{dx} = \frac{d}{dx}(x^T \mu)$$

$$S \frac{\sigma \frac{d\sigma}{dx}}{\sigma} + \sigma \frac{dS}{dx} = \frac{dr}{dx}$$

$$\frac{dr}{dx} = S \frac{d\sigma}{dx} + \sigma \frac{dS}{dx} \qquad (3)$$

Equation (3) represents the Indifferent Sharpe Lines (ISLs) and setting $\frac{dS}{dx} = 0$ results in the Optimal Sharpe Line (OSL).

The first order condition for optimizing the portfolio Sharpe ratio, subject to the budget constraint is,

$$\frac{1}{(x^T V x)^{1/2}} \frac{d}{dx}(x^T \mu - r_f) - \frac{x^T \mu - r_f}{x^T V x} \frac{d}{dx}(x^T V x)^{1/2} + \lambda \frac{d}{dx}(1 - x^T I) = 0$$

$$\frac{1}{(x^T V x)^{1/2}} \frac{d}{dx}(x^T \mu) = \frac{x^T \mu - r_f}{x^T V x} \frac{d}{dx}(x^T V x)^{1/2} - \lambda \frac{d}{dx}(x^T I)$$

$$\mu = \frac{x^T \mu - r_f}{(x^T V x)^{1/2}} \frac{d}{dx}(x^T V x)^{1/2} - (x^T V x)^{1/2} \lambda \frac{d}{dx}(x^T I)$$

$$\mu = S \frac{d\sigma}{dx} - \sigma \lambda I \qquad (4)$$

Equation (4) represents the Implied Returns (IRs). When the budget constraint is non-binding, the Lagrange multiplier λ is 0. The implied returns are reduced by $\sigma\lambda$ when the budget constraint is binding.

Note that we have used the discrete versions of OSL, ISLs and IRs in the paper to facilitate practitioner wanting to implement the framework presented.

REFERENCES

Agarwal, V., Naik, N. Y., 2004. Risk and Portfolio Decisions Involving Hedge Funds. Review of Financial Studies 17 (1), 63–98

Dowd, K., 2000. Adjusting for Risk: An Improved Sharpe Ratio. International Review of Economics and Finance 9 (3), 209–222

Defined Benefit Plans amid Market Volatility, CFO Research Services in collaboration with Towers Perrin, September 2008

Eling, Martin and Schuhmacher, Frank, Does the Choice of Performance Measure Influence the Evaluation of Hedge Funds?. Journal of Banking and Finance, 31(9), pp. 2632-2647

Gregoriou, G. N., Gueyie, J.-P., 2003. Risk-Adjusted Performance of Funds of Hedge Funds Using a Modified Sharpe Ratio. Journal of Alternative Investments 6 (Winter), 77–83

Markowitz, Harry M., Portfolio Selection, The Journal of Finance, 7(1), March, pp. 77-91

Opdyke, J.D., Comparing Sharpe Ratios: So Where are the P-Values? Journal of Asset Management, Vol. 8, No. 5, pp. 308-336

Sharpe, William F. "Imputing Expected Security Returns from Portfolio Composition." The Journal of Financial and Quantitative Analysis, Vol. 9, No. 3, pp. 463-472

Sharpe, William F. "Mutual Fund Performance." Journal of Business*,* January 1966, pp. 119-138.

Simons, Neil, September 2008, "Risk and Return Properties of Alternative Betas." Internal White Paper, Northwater Capital Management